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On Non - Homogeneous Cubic Equation With Four Unknowns $x^2 + y^2 + 4(35z^2 - 4 - 35w^2) = 6xyz$

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ABSTRACT

This paper is devoted to obtain non-zero distinct integer solutions to non-homogeneous cubic equation with four unknowns represented by $x^2 + y^2 + 4(35z^2 - 4 - 35w^2) = 6xyz$ along with few observations.

KEY WORDS: NON-HOMOGENEOUS, CUBIC WITH FOUR UNKNOWNS, INTEGER SOLUTIONS 2010 MATHEMATICS SUBJECT CLASSIFICATION: 11D09.

INTRODUCTION

The cubic Diophantine equations are rich in variety and offer an unlimited field for research. This paper concerns with another interesting cubic Diophantine equation with four unknowns $x^2 + y^2 + 4(35z^2 - 4 - 35w^2) = 6xyz$ for determining its infinitely many non-zero integral solutions.

Notations Used:

1. Regular Polygonal Number of rank n with sides m : $t_{m,n} = n[1 + \frac{(n-1)(m-1)}{2}]$
2. Pyramidal Number of rank n with sides m : $p_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)]$
3. Pronic Number of rank n : $pr_n = n(n+1)$
4. Stella Octangular Number of rank n: $SO_n = n(2n^2 - 1)$
5. Octahedral Number of rank n: $OH_n = \frac{1}{3}n(2n^2 + 1)$
6. Star Number of rank n : $S_n = 6n(n-1) + 1$
7. Pentatope Number of rank n : $pt_n = \frac{n(n+1)(n+2)(n+3)}{24}$

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Method of Analysis: The homogeneous cubic equation with four unknowns to be solved is

$$x^2 + y^2 + 4(35z^2 - 4 - 35w^2) = 6xyz \quad (1)$$

Introducing the linear transformations

$$x = 2X + 12z, y = 4 \quad (2)$$

in (1), it leads to

$$X^2 = z^2 + 35w^2 \quad (3)$$

We present below different methods of solving (3) and thus, obtain different patterns of integral solutions to (1).

Pattern-I

It is observed that (3) is satisfied by

$$w = 2rs, z = 35r^2 - s^2, X = 35r^2 + s^2 \quad (4)$$

Hence, in view of (2) and (4), the non- zero integral solutions of (1) are given by

$$x = x(r,s) = 490r^2 - 10s^2$$
$$y = y(r,s) = 4$$
$$z = z(r,s) = 35r^2 - s^2$$
$$w = 2rs$$

Properties:

1. $x(r(r+1),r)-35z(r(r+1),r)+w(r(r+1),r)=25t_{4,r}+4P_r^5$
2. $x(r(r+1),r+2)-35z(r(r+1),r+2)+y(r(r+1),r+2)=25(\Pr_r)^2+12P_r^3+4$
3. $35\{x(r,r)-10z(r,r)\}$ is a perfect square.
4. $6\{x(r,s)-35z(r,s)\}$ is a Nasty Number.
5. $x(r,3(r-1))-35z(r,3(r-1))+y(r,3(r-1))+w(r,3(r-1))=75t_{4,r-1}+S_r+3$

Pattern-II
Method of factorization

Write (3) as $z^2+35w^2=X^2=X^2*1$ (5)

Assume $X=a^2+35b^2$ (6)

where a and b are non-zero integers.

Write 1 as $1=\frac{(1+i\sqrt{35})(1-i\sqrt{35})}{6^2}$ (7)

Using (5), (6) in (5) and applying the method of factorization, define

$$(z+i\sqrt{35}\,w)=\frac{1}{6}(1+i\sqrt{35})(a+i\sqrt{35}\,b)^2$$

Equating the real and imaginary parts, we have

$$z=\frac{1}{6}[a^2-70ab-35b^2]$$

$$w=\frac{1}{6}[a^2+2ab-35b^2]$$

As our interest centres on finding integer solutions, it seen that the values of z, w, X are integers, when a= 7 A, B = 7B

Thus, we have

$$\left.\begin{array}{l} X=X(A,B)=36A^2+1260\,AB \\ z=z(A,B)=6A^2-210\,B^2-420\,AB \\ w=z(A,B)=6A^2-210\,B^2+12\,AB \end{array}\right\} \quad (8)$$

Hence, in view of (2) and (8), the non- zero integral solutions of (1) are found to be

$$x=x(A,B)=144A^2-5040AB$$
$$y=y(A,B)=4$$
$$z=z(A,B)=6A^2-210B^2-420AB$$
$$w=z(A,B)=6A^2-210B^2+12AB$$

Properties:

1. $x(A,A+1)-24z(A,A+1)=1260t_{4,A+1}+10080t_{3,A}$
2. $w(A(A+1),(A+2)(A+2))-z(A(A+1),(A+2)(A+2))=129\,pt_A$
3. $w(A,A)-Z(A,A)$ is a Nasty Number.
4. $z(A,B)+w(A,B)\equiv 0(\text{mod}12)$
5. $x(A,(A+1))+12[z(A,(A+1))+w(A,(A+1))]+420t_{4,A+1}+9936\Pr_A$

Note: It is worth mentioning here that, 1 may also be represent as the product of complex conjugates as shown below:

$$1=\frac{(17+i\sqrt{35})(17-i\sqrt{35})}{324}$$

$$1=\frac{(13+i3\sqrt{35})(13-i3\sqrt{35})}{484}$$

$$1=\frac{(35r^2-s^2+2i\sqrt{35}rs)(35r^2-s^2-2i\sqrt{35}rs)}{(35r^2+s^2)^2}$$

Pattern-III
Method of factorization

Rewrite (3) as

$$X^2-35w^2=z^2*1 \quad (9)$$

Let $z=p^2-35q^2$ (10)

where p and q are non-zero integers.

Write 1 as $1=(6+\sqrt{35})(6-\sqrt{35})$ (11)

Substituting (10) and (11) in (9) and employing the method of factorization, define

$(X+\sqrt{35}\,w)=(6+\sqrt{35})(p+i\sqrt{35}\,q)^2$ Equating the rational and irrational parts, we have

$$X=X(p,q)=6p^2+210q^2-70pq$$
$$w=w(p,q)=p^2+35q^2-12pq$$

Thus, the corresponding non- zero distinct integral solutions of (1) are

$$x=x(p,q)=24p^2-140pq$$
$$y=y(p,q)=4$$
$$z=z(p,q)=p^2-35q^2$$
$$w=w(p,q)=p^2+35q^2-12pq$$

Properties:

1. $x(p(p+1),2p+1)-24z(p(p+1),2p+1)+840P_p^4=840t_{4,2p+1}$
2. $2\{z(p,(2p^2-1))+w(p,(2p^2-1))+24SO_p\}$ is a perfect square.
3. $35\{w(p,(2p^2+1))-z(p,(2p^2+1))+36OH_p\}$ is a Nasty Number.
4. $x(p,(p+1)(p+2))-12z(p,(p+1)(p+2))+12w(p,(p+1)(p+2))=24P_p^3$

Pattern-IV **System of double equations**

Observe that (3) as $X^2-z^2=35w^2$

It can be represented as the system of double equation as shown below:

Table 1. System of double equations

System	1	2	3	4	5	6
$X+z$	$35w^2$	$7w^2$	$5w^2$	w^2	$35w$	$7w$
$X-z$	1	5	7	35	w	$5w$

System :1	System:2	System:3
$x = 980T^2 + 980T + 240$ $y = 4$ $z = 70T^2 + 70T + 17$ $w = 2T + 1$	$x = 196T^2 + 196T + 24$ $y = 4$ $z = 14T^2 + 14T + 1$ $w = 2T + 1$	$x = 140T^2 + 140T$ $y = 4$ $z = 10T^2 + 10T - 1$ $w = 2T + 1$
System :4	**System:5**	**System:5**
$x = 28T^2 + 28T - 168$ $y = 4$ $z = 2T^2 + 2T - 17$ $z = 2T + 1$	$x = 240T$ $y = 4$ $z = 17T$ $w = T$	$x = 24T$ $y = 4$ $z = T$ $w = T$

Remarkable Observations:

- If the non-zero integer quadrup $(x_0, 4, z_0, w_0)$ is any solution of (1), then, each of the following three quadruples of integers based on x_0, z_0 and w_0 also satisfies (1).

Quadruple:1 (x_n, y_n, z_n, w_n)

$$x_n = \frac{1}{2}\left[-[38\times 4^{n-1} + 54(-4)^{n-1}]x_0 + [114\times 4^{n-1} + 18(-4)^{n-1}]z_0\right]$$

$$y_n = 4$$

$$z_n = \frac{1}{2}\left[-3[4^{n-1} + (-4)^{n-1}]x_0 + [9\times 4^{n-1} + (-4)^{n-1}]z_0\right]$$

$$w_n = 4^n w_0$$

Quadruple:2 (x_n, y_n, z_n, w_n)

$$x_n = 2(6)^{2n}x_0 + 6\left[[70\times 36^{n-1} - 2(-36)^{n-1}]z_0 + [-70\times 36^{n-1} + (-36)^{n-1}]w_0\right]$$

$$y_n = 4$$

$$z_n = [35\times 36^{n-1} - (-36)^{n-1}]z_0 + [-35(36^{n-1} + (-36)^{n-1})]w_0$$

$$w_n = [-36^{n-1} - (-36)^{n-1}]z_0 + [36^{n-1} - 35(-36)^{n-1})]w_0$$

Quadruple:3 (x_n, y_n, z_n, w_n)

$$x_n = \left|-70 + 72(-1)^n]x_0 + [420(1-(-1)^n]w_0\right| + 12z_0$$

$$y_n = 4$$

$$z_n = z_0$$

$$w_n = -6[1-(-1)^n]x_0 + [36 - 35(-1)^n]w_0$$

CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non - homogeneus cubic equation with four unknowns. As the cubic equations are rich in variety, one may search for other forms of cubic equations with multi-variables to obtain their corresponding solutions.

REFERENCES

G. Janaki, C. Saranya, "Integral solutions of the ternary cubic equation $3(x^2+y^2)-4xy+2(x+y+1)=972z^3$", IRJET, Vol.04, Issue 3, March 2017, 665-669.

L.E. Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing company, NewYork, 1952.

L.E. Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing company, NewYork, 1952.

L.J. Mordell, Diophantine equations, Academic press, New York, 1969.

M.A. Gopalan, B. Sivakami, "Integral solutions of the ternary cubic equation $4x^2-4xy+6y^2=((k+1)^2+5)w^3$",Impact J.Sci.Tech, Vol.6, No.1, 2012, 15-22.

M. A. Gopalan, G. Sangeetha, "On the ternary cubic Diophantine equation $y^2=Dx^2+z^3$", Archimedes J.Math 1(1), 2011, 7-14.

M.A. Gopalan, B. Sivakami, "On the ternary cubic Diophantine equation $2xz=y^2(x+z)$", Bessel J.Math 2(3), 2012, 171-177.

M.A. Gopalan, K. Geetha, "On the ternary cubic Diophantine equation $x^2+y^2-xy=z^3$", Bessel J.Math., 3(2), 2013,119-123.

M.A. Gopalan, S. Vidhyalakshmi, A.Kavitha "Observations on the ternary cubic equation $x^2+y^2+xy=12z^3$", Antartica J.Math 10(5), 2013, 453-460.

M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi, "Lattice points on the non-homogeneous cubic equation $x^3+y^3+z^3+(x+y+z)=0$", Impact J.Sci. Tech, Vol.7, No.1, 2013, 21-25.

M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi "Lattice points on the non-homogeneous cubic equation $x^3+y^3+z^3-(x+y+z)=0$", Impact J.Sci. Tech, Vol.7, No1, 2013, 51-55,

M.A. Gopalan, S. Vidhyalakshmi, S. Mallika, "On the ternary non-homogenous cubic equation $x^3+y^3-3(x+y)=2(3k^2-2)z^3$", Impact J.Sci. Tech, Vol.7, No.1, 2013, 41-45.

M.A. Gopalan, N. Thiruniraiselvi and V. Kiruthika, "On the ternary cubic diophantine equation $7x^2-4y^2=3z^3$", IJRSR, Vol.6, Issue-9, Sep-2015, 6197-6199.

M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, J. Maheswari, "On ternary cubic diophantine equation $3(x^2+y^2)-5xy+x+y+1=12z^3$", International Journal of Applied Research, 1(8), 2015, 209-212.

M.A. Gopalan, S. Vidhyalakshmi, G. Sumathi, "On the homogeneous cubic equation with four unknowns $X^3+Y^3=14Z^3-3W^2(X+Y)$", Discovery, 2(4), 2012, 17-19.

M.A. Gopalan, S. Vidhyalakshmi, E. Premalatha, C. Nithya, "On the cubic equation with four unknowns $x^3+y^3=13(k^2+3s^2)zw^2$", IJSIMR, Vol.2, Issue 11, Nov-2014, 923-926.

M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, "On the cubic equation with four unknowns $x^3+4z^3=y^3+4w^3+6(x-y)^3$", International Journal of Mathematics Trends and Technology, Vol 20, No.1, April 2015, 75-84.

M.A.Gopalan, E.Premalatha, N.Uma Maheshwari, "On the homogeneous cubic Diophantine equation with four unknowns $3(x^3+y^3)=8zw^2$", Alochana Chakra Journal, Vol.IX, Issue V, May 2020, 285-291.

R. Anbuselvi, K. Kannaki, "On ternary cubic diophantine equation $3(x^2+y^2)-5xy+x+y+1=15z^3$", IJSR, Vol.5, Issue-9, Sep 2016, 369-375.

R. Anbuselvi, K.S. Araththi, "On the cubic equation with four unknowns $x^3+y^3=24zw^2$", IJERA, Vol.7, Issue 11 (Part-I), Nov-2017, 01-06.

S. Vidhyalakshmi, M.A. Gopalan, A. Kavitha, "Observation on homogeneous cubic equation with four unknowns $X^3+Y^3=7^{2n}ZW^2$", IJMER, Vol.3, Issue 3, May-June 2013, 1487-1492.

S. Vidyalakshmi, T.R. Usharani, M.A. Gopalan, "Integral solutions of non-homogeneous ternary cubic equation $ax^2+by^2=(a+b)z^3$", Diophantus J.Math 2(1), 2013, 31-38.

S. Vidhyalakshmi, M.A. Gopalan, S. Aarthy Thangam, "On the ternary cubic Diophantine equation $4(x^2+x)+5(y^2+2y)=-6+14z^3$" International Journal of Innovative Research and Review (JIRR), Vol 2(3)., pp 34-39, July-Sep 2014.